Possible Solutions

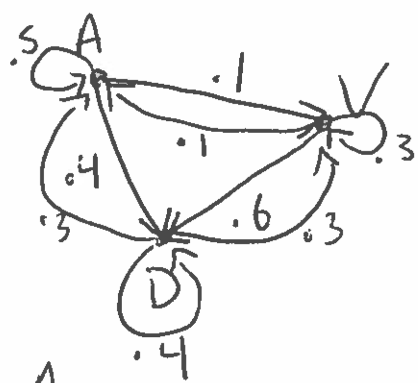
Graph Theory
Group Work

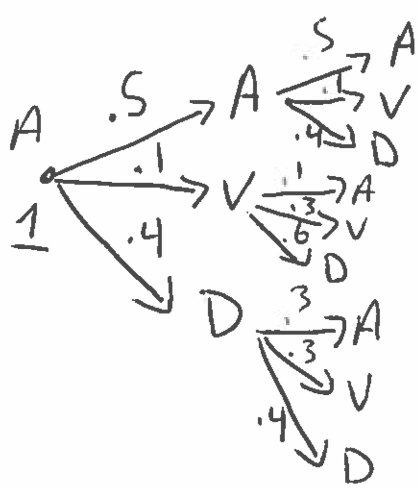
Names:

A car rental company has three locations in Mexico City: the International Airport, Oficina Vallejo, and Downtown. Customers can drop off their vehicles at any of these locations. Based on prior experience, the company expects that, at the end of each day, 40% of the cars that begin the day at the Airport will end up Downtown, 50% will return to the Airport, and 10% will be at Oficina Vallejo. Similarly, 60% of the Oficina Vallego cars will end up Downtown, with 30% returning to Oficina Vallejo and 10% at the airport. Finally, 30%of the Downtown cars will end up at each of the other locations, with 40% staying at the Downtown location.

Question. Model this situation with a directed network. If the company starts with all the cars at the Airport, how will the cars be distributed after two days of rentals?

A= Airport
V= Oficina Vallejo
D= Downtown





On day 2,

$A = (.5)(.5) + (.1)(.1) + (.4)(.3) = .38$

$V = (.5)(.1) + (.1)(.3) + (.4)(.3) = .2$

$D = (.5)(.4) + (.1)(.6) + (.4)(.4) = .42$

Further Question. If all the cars start at the Downtown location, approximately how will the cars be distributed after a year? In a year, would the fact that they all started at the Downtown location make a difference?

It doesn't matter! Will be about $A = .316$, $V = .237$, $D = .447$.

Possible Solutions

Graph Theory
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Names:

Let $G = (V, E)$ be a simple, connected, undirected graph, and let

$$P = \{\{v_1, v_2\} | v_1, v_2 \in V,\ v_1 \neq v_2\}$$

be the set of all unordered pairs of vertices. Define a function $f : E \to P$ as follows: If $e \in E$ is an edge in G, then $f(e) = \{a, b\}$, where a and b are the vertices that e touches.

Question.

(a) Explain why $f(e)$ is always a set of size 2; i.e. show that f is well-defined.

(b) Is f one-to-one? Prove or disprove. If not, when is it one-to-one?

(c) Is f onto? Prove or disprove. If not, when is it onto?

(d) How would this construction change if you allowed for directed graphs?

(a) Every edge is connected to 2 vertices.

(b) Yes because there are no multi-edges; i.e. every pair of vertices can share exactly one edge.

(c) No, unless G is a complete graph.

(d) We would need to let $P = \{(v_1, v_2) \in V \times V \mid v_1 \neq v_2\}$ ~~but~~ and the function f would need to specify the order of the edge.

Further Question. Can you think of a scenario where it would be desirable for each edge to have more than 2 vertices that it touches (say, for instance, 3 vertices)? Generalize the construction above to account for this scenario.

Anytime we want to consider connections of more than 2 vertices. We would let $P = \{\{v_1, v_2, v_3 \mid v_1, v_2, v_3 \in V$ are pairwise disjoint$\}$

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Graph Theory
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Names:

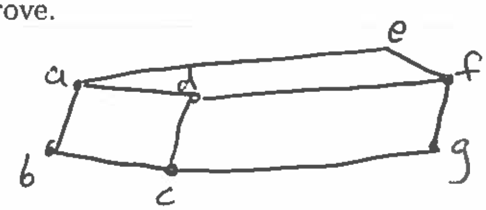
Let $G = (V, E)$ be a connected, undirected graph. Define a relation R on V as follows: for any vertices $a, b \in V$, $a \; R \; b$ if there is a path from a to b with an even number of edges. (A path may use the same edge twice)

Question.

(a) Prove that R is an equivalence relation.

(b) Find the equivalence classes for the graph below.

(c) What if a path could not use the same edge twice. Would R still be an equivalence relation? Prove or disprove.



(a) Reflexive: The 0-length path is even, so $vRv \ \forall v \in V$. ✓

Symmetric: If $v_0 e_1 v_1 \cdots v_{n-1} e_n v_n$ is an even path from v_0 to v_n then $v_n e_n v_{n-1} \cdots v_1 e_1 v_0$ is an even path from v_n to v_0.

So $uRv \Rightarrow vRu \ \forall u,v \in V$. ✓

This part is actually tricky → Transitive: If $v_0 R v_1$ and $v_1 R v_2$ then then are even paths P_1 and P_2 from v_0 to v_1 and v_1 to v_2, respectively. Since even + even = even, $P_2 P_1$ is an even path from v_0 to v_2. ✓

(b) $[a] = \{a, c, f\}$ and $[b] = \{b, d, g, e\}$.

(c)

Further Question. If the graph were allowed to be directed, would R still be an equivalence relation? What if it were unconnected? Can you think of a scenario where this type of equivalence relation could be used?

If directed, Symmetry would fail.
Unconnected is fine.

Possible Solutions

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Names:

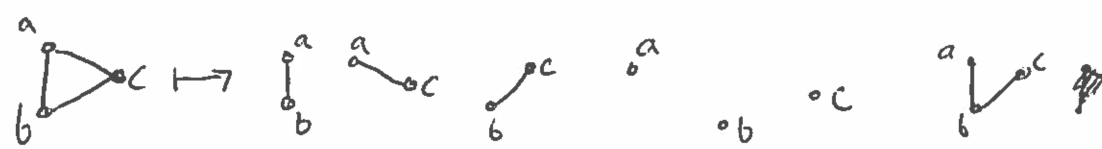
Let $G = (V, E)$ be a graph. A *subgraph*, H, of G is a graph with vertex set $V_H \subseteq V$ and edge set $E_H \subseteq E$.

Question.

(a) Write down all the subsets of K_3, the complete graph on three vertices.

(b) Let $V_H \subseteq V$ and $E_H \subseteq E$. Is there always a subgraph of G with vertex set V_H and edge set E_H? Explain.

(c) Let G be a connected, undirected graph. Prove that there is a subgraph T of G such that T contains all the vertices of G, and T is a tree. (Such a subgraph is called a *spanning tree*.) Give a constructive proof that explains how to construct a spanning tree of a graph.

(a)  and so on...

(b) No. If there is some $e \in E_H$ which touches some $v \in V \setminus V_H$ then (V_H, E_H) is not a subgraph.

(c) Start with a vertex $r \in V$ to make the root of the tree.

(i) Find a vertex which has done been grabbed yet which is connected to your current vertex and draw an edge to it.

(ii) If there is none, go backwards until there is one.

Repeat (i) and (ii).

Once you get back to the root and it has no more uncollected neighbors, you are done.

Further Question. Is there more than one spanning tree? Can you say exactly how many there will be? Yes. This is a hard question.

Possible Solutions

Graph Theory
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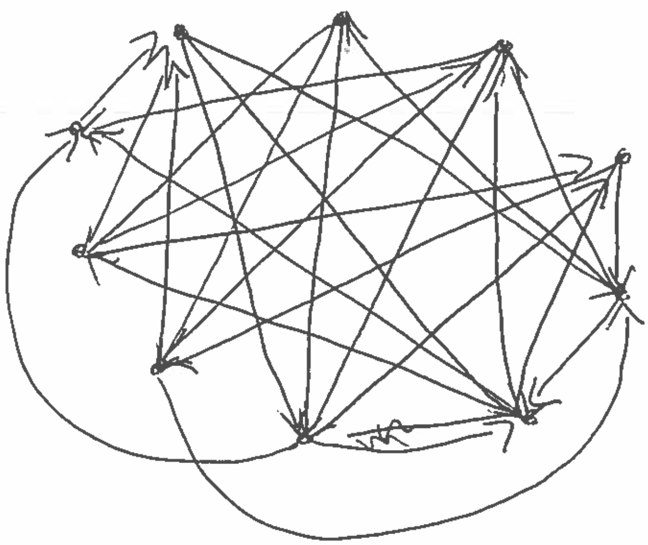
Names:

Suppose you are organizing a soccer league which has ten teams, each with a different home field. You want to schedule each team for six games (3 home and 3 away) with each game being against a different opponent.

Question. Is this possible? Explain how you would model it (if it is possible) and give a model if it is possible. Explain. How many possible configurations are there?



This is a mess. So we have to try something other than just trial and error.

We want a simple directed graph on 10 vertices each with 3 out degree and 3 indegree. Also we don't want (u,v) and (v,u) to both be edges for any $u,v \in V$. Let's number the teams $1, 2, \ldots, 10$.

For each team $i \in \{1, \ldots, 10\}$, let i play $i+1$, $i+2$ and $i+3$ mod 10. Then these conditions are satisfied.

Lots! How ever many, it must be even.

Further Question. Suppose, in addition, that you want to make each team travel the same amount (or as close to as possible). How would you change your model? Suppose that you also want each team to play four games on Saturday and only two on Sunday due to time constraints. How would you change your model?

You would need to add weights to your model.

You would possibly need to add a Saturday and Sunday vertex